Numerical Integration – Prof. Richard B. Goldstein

Estimate the Definite Integral: $J = \int_a^b f(x)\,dx$

TRAPEZOID $T_n = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n) + E_n$ where $h = \frac{b-a}{n}$, $f_k = f(x_k), x_k = a + kh$

$$E_n = -\frac{(b-a)h^2}{12} f''(c) \approx \frac{|T_n - T_{n/2}|}{3}$$

MIDPOINT $M_n = h(f_{0.5} + f_{1.5} + f_{2.5} + \cdots + f_{n-1.5} + f_{n-0.5}) + E_n$

$$E_n = \frac{(b-a)h^2}{24} f''(c) \approx \frac{|M_n - M_{n/2}|}{3}$$

SIMPSON $S_n = \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n) + E_n$

$$E_n = -\frac{(b-a)h^4}{180} f^{(4)}(c) \approx \frac{|S_n - S_{n/2}|}{15}$$

GAUSS $G_n = \frac{b-a}{2}\sum_{k=1}^{n} w_k f\left[\left(\frac{b-a}{2}\right)t_k + \left(\frac{b+a}{2}\right)\right]$

n = 2: $w_1 = w_2 = 1$, $t_1 = 0.57735\ 02692$, $t_2 = -t_1$

n = 3: $w_1 = w_3 = 5/9$, $w_2 = 8/9$, $t_1 = 0.77459\ 66692$, $t_2 = 0$, $t_3 = -t_1$

Error Estimate: $R_n = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(c)$

ROMBERG $R_{1,1} = T_1,\ R_{2,1} = T_2,\ R_{3,1} = T_4,\ R_{4,1} = T_8, \ldots$

and $R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$

Table:

$R_{1,1}$			
$R_{2,1}$	$R_{2,2}$		
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$	
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$

Error Estimate: $| R_{n,n} - R_{n-1,n-1} |$

IMPROPER INTEGRALS

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x)}{(x-a)^p}\,dx = \int_a^b \frac{g(x) - P_4(x)}{(x-a)^p}\,dx + \int_a^b \frac{P_4(x)}{(x-a)^p}\,dx$$

where the first integral is estimated by Simpson's rule with h = (b - a)/4 (<u>note</u>: assume 0 at the left endpoint x = a) and the second is exactly

$$\sum_{k=0}^{4} \frac{g^{(k)}(a)}{k!(k+1-p)}(b-a)^{k+1-p}$$