Numerical Integration – Prof. Richard B. Goldstein

Estimate the Definite Integral

$$J = \int_a^b f(x)\,dx \quad \text{Let } h = (b-a)/n \text{ or } b = a + nh$$

TRAPEZOID

$$T_n = h[f(a) + 2f(a+h) + 2f(a+2h) + \ldots + 2f(a+(n-1)h) + f(a+nh)]/2$$

$$E_n = -\frac{(b-a)h^2}{12} f''(c) \approx \frac{|T_n - T_{n/2}|}{3}$$

MIDPOINT

$$M_n = h[f(a+0.5h) + f(a+1.5h) + f(a+2.5h) + \ldots + f(a+(n-0.5)h)]$$

$$E_n = \frac{(b-a)h^2}{24} f''(c) \approx \frac{|M_n - M_{n/2}|}{3}$$

SIMPSON

$$S_n = h[f(a) + 4f(a+h) + 2f(a+2h) + \ldots + 2f(a+(n-1)h) + 4f(a+(n-1)h) + f(a+nh)]/3$$

$$E_n = -\frac{(b-a)h^4}{180} f^{(4)}(c) \approx \frac{|S_n - S_{n/2}|}{15}$$

ROMBERG

$$R_{1,1} = T_1,\ R_{2,1} = T_2,\ R_{3,1} = T_4,\ R_{4,1} = T_8, \ldots$$

and $R_{k,j} = R_{k,j-1} + \dfrac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}$

Table:

$R_{1,1}$			
$R_{2,1}$	$R_{2,2}$		
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$	
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$

Error Estimate: $| R_{n,n} - R_{n-1,n-1} |$

GAUSS

$$G_n = \tfrac{b-a}{2} \sum_{k=1}^{n} w_k f\left[\left(\tfrac{b-a}{2}\right)t_k + \left(\tfrac{b+a}{2}\right)\right]$$

n = 2: $w_1 = w_2 = 1$, $t_1 = 0.57735\ 02692$, $t_2 = -t_1$

n = 3: $w_1 = w_3 = 5/9$, $w_2 = 8/9$, $t_1 = 0.77459\ 66692$, $t_2 = 0$, $t_3 = -t_1$

Error Estimate: $R_n = \dfrac{2^{2n+1}(n!)^4}{(2n+1)\left[(2n)!\right]^3} f^{(2n)}(c)$

IMPROPER INTEGRALS

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x)}{(x-a)^p}\,dx = \int_a^b \frac{g(x) - P_4(x)}{(x-a)^p}\,dx + \int_a^b \frac{P_4(x)}{(x-a)^p}\,dx$$

where the first integral is estimated by Simpson's rule with h = (b - a)/4 (note: assume 0 at the left endpoint x = a) and the second is exactly

$$\sum_{k=0}^{4} \frac{g^{(k)}(a)}{k!(k+1-p)}(b-a)^{k+1-p}$$

Example: $\int_2^6 \frac{1}{x} dx = \ln(6) - \ln(2) \approx 1.098612$ h = (6 – 2)/4 = 1 or h = (6 – 2)/8 = 0.5

Trapezoid: T_4 = 1[(1/2) + 2(1/3) + 2(1/4) + 2(1/5) + 1/6]/2 = 2.233333/2 = 1.116667

T_8 = 0.5[(1/2) + 2(1/2.5) + 2(1/3) + 2(1/3.5) + … + 2(1/5.5) + 1/6]/2
= 0.5[4.412827]/2 = 1.103211

error estimate ≈ |1.103211 – 1.116667|/3 = 0.004485 (actual 0.004599)

Midpoint: M_4 = 1[1/2.5 + 1/3.5 + 1/4.5 + 1/5.5] =1.089755

M_8 = 0.5[1/2.25 + 1/2.75 + 1/3.25 + … + 1/5.25 + 1/5.75] = 1.096325

error estimate ≈ |1.096325 – 1.089755|/3 = 0.002190 (actual 0.002287)

Simpson: S_4 = 1[(1/2) + 4(1/3) + 2(1/4) + 4(1/5) + 1/6]/3 = 3.3/3 = 1.100000

S_8 = 0.5[(1/2) + 4(1/2.5) + 2(1/3) + 4(1/3.5) + … + 4(1/5.5) + (1/6)]/3
= 0.5 (6.592352)/3 = 1.098725

error estimate ≈ |1.098725 – 1.100000|/15 = 0.000085 (actual 0.000113)

Comparisons for n = 64

Trapezoid	T_{64} = 1.098 684 619	error est. = 0.000 072 299
Midpoint	M_{64} = 1.098 576 127	error est. = 0.000 036 135
Simpson	S_{64} = 1.098 612 320	error est. = 0.000 000 031