Numerical Integration – Romberg/Gauss - Prof. Richard B. Goldstein

Estimate the Definite Integral $J = \int_a^b f(x)\,dx$ Let h = (b – a)/n or b = a + nh

ROMBERG $T_n = h[f(a) + 2f(a + h) + 2f(a + 2h) + \ldots + 2f(a + (n-1)h) + f(a + nh)]/2$

$R_{1,1} = T_1$
$R_{2,1} = T_2$
$R_{3,1} = T_4$
$R_{4,1} = T_8$

$R_{2,2} = [4R_{2,1} - R_{1,1}]/3$
$R_{3,2} = [4R_{3,1} - R_{2,1}]/3$
$R_{4,2} = [4R_{4,1} - R_{3,1}]/3$

$R_{3,3} = [16R_{3,2} - R_{2,2}]/15$
$R_{4,3} = [16R_{4,2} - R_{3,2}]/15$

$R_{4,4,} = [64R_{4,3} - R_{3,3}]/63$

Table:

$R_{1,1}$			
$R_{2,1}$	$R_{2,2}$		
$R_{3,1}$	$R_{3,2}$	$R_{3,3}$	
$R_{4,1}$	$R_{4,2}$	$R_{4,3}$	$R_{4,4}$

Error Estimate: $|R_{n,n} - R_{n-1,n-1}|$

GAUSS $G_n = \frac{b-a}{2}\sum_{k=1}^{n} w_k f\left[\left(\frac{b-a}{2}\right)t_k + \left(\frac{b+a}{2}\right)\right]$ Error Estimate: $R_n = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(c)$

Let h = (b - a)/2 be half the interval width and m = (b + a)/2 be its midpoint

$G_2 = h[f(m - 0.57735h) + f(m + 0.57735h)]$

Error = $7.407 \times 10^{-3} f^{(4)}(c)$

$G_3 = h[0.55556f(m - 0.77460h) + 0.88889f(m) + 0.55556f(m + 0.77460h)]$

Error = $6.349 \times 10^{-5} f^{(6)}(c)$

$G_4 = h[0.34785f(m - 0.86114h) + 0.65215f(m - 0.33998h) +$
$0.65215f(m + 0.33998h) + 0.34785f(m + 0.86114h)]$

Error = $2.880 \times 10^{-7} f^{(8)}(c)$

Example: $\int_{2}^{6} \frac{1}{x} dx = \ln(6) - \ln(2) \approx 1.098612$

Romberg: $R_{1,1} = T_1 = 4[(1/2) + (1/6)]/2 = 1.333\,333$

$R_{2,1} = T_2 = 2[(1/2) + 2(1/4) + (1/6)]/2 = 1.166\,667$

$R_{3,1} = T_4 = 1[(1/2) + 2(1/3) + 2(1/4) + 2(1/5) + 1/6]/2 = 2.233333/2 = 1.116\,667$

$R_{4,1} = T_8 = 0.5[(1/2) + 2(1/2.5) + 2(1/3) + 2(1/3.5) + \ldots + 2(1/5.5) + 1/6]/2$

$= 0.5[4.412827]/2 = 1.103\,211$

$R_{2,2} = [4(1.166667) - 1.333333]/3 = 1.111\,112$

$R_{3,2} = [4(1.116667) - 1.166667]/3 = 1.100\,000$

$R_{4,2} = [4(1.103211) - 1.116667]/3 = 1.098\,726$

$R_{3,3} = [16(1.100000) - 1.111112]/15 = 1.099\,259$

$R_{4,3} = [16(1.098726) - 1.100000]/15 = 1.098\,641$

$R_{4,4} = [64(1.098641) - 1.099259]/63 = 1.098\,631$

error est. = |1.098 631 – 1.099 259| = 0.000 628 (actual 0.000 029)

Gauss: $h = (6 - 2)/2 = 2$, $m = (6 + 2)/2 = 4$

$G_2 = 2[1/[4 - 0.577350(2)] + 1/[4 + 0.577350(2)]$

$= 2[0.351\,457 + 0.193\,998] = 1.090\,909$ (error 0.007 703)

$G_3 = 2[0.555556(1/\{4 - 0.774597(2)\}) + 0.888889(1/4) + 0.555556(1/\{4 + 0.774597(2)\})]$

$= 2[0.226\,683 + 0.222\,222 + 0.100\,115]$

$= 1.098\,040$ (error 0.000 572)

$G_4 = 2[0.347855(1/\{4 - 0.861136(2)\}) + 0.652145(1/\{4 - 0.339981(2)\}) +$

$0.652145(1/\{4 + 0.339981(2)\}) + 0.347855(1/\{4 + 0.861136(2)\})]$

$= 2[0.152\,720 + 0.196\,427 + 0.139\,348 + 0.060\,790]$

$= 1.098\,570$ (error 0.000 042)

Comparisons

Trapezoid (n = 8)	error = 0.004 599
Midpoint (n = 8)	error = 0.002 287
Simpson (n = 8)	error = 0.000 113
Romberg (n = 8)	error = 0.000 029
Gauss (n = 8)	error = 0.000 001