IMPROPER INTEGRALS USING NUMERICAL ANALYSIS

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Types:

Left Singularity: $\displaystyle\int_a^b \frac{g(x)}{(x-a)^p}\,dx$ where $0 < p < 1$

Interior Singularity: $\displaystyle\int_a^b = \int_a^{c-} + \int_{c+}^b$ let t = -x for first integral to have a singularity on the left

Semi-infinite: $\displaystyle\int_a^\infty$ let t = 1/x → $\displaystyle\int_0^{1/a}$

Double-infinite: $\displaystyle\int_{-\infty}^\infty = \int_{-\infty}^a + \int_a^\infty$ and use t =1/x in each half

Text Method – extraction

Assume $g \in \mathcal{C}^5[a, b]$. We expand g(x) by a Taylor Series about x = a:

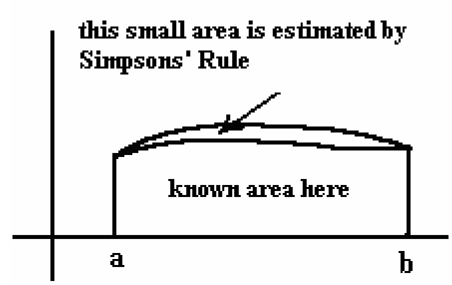
$$P_4(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \frac{g'''(a)}{3!}(x-a)^3 + \frac{g^{(4)}(a)}{4!}(x-a)^4$$

and rewrite $\displaystyle\int_a^b f(x)\,dx = \int_a^b \frac{g(x)}{(x-a)^p}\,dx = \int_a^b \frac{g(x)-P_4(x)}{(x-a)^p}\,dx + \int_a^b \frac{P_4(x)}{(x-a)^p}\,dx$

where the first integral is estimated by Simpson's rule with h = (b - a)/4 or h = (b – a)/6. Note, that one should assume 0 at the left endpoint x = a. The second integral is found exactly to be:

$$\sum_{k=0}^{4} \frac{g^{(k)}(a)}{k!(k+1-p)}(b-a)^{k+1-p}$$

note: any extraction will help



Example #1: $\int_0^1 \frac{e^{2x}}{\sqrt[5]{x^2}}dx$

Version #1

$p = 2/5 \; g(x) = e^{2x}$

$$P_4(x) = 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \frac{(2x)^4}{24} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3}$$

$$J_2 = \int_0^1 \frac{P_4(x)}{x^{2/5}}dx = \frac{1}{3/5} + \frac{2}{8/5} + \frac{2}{13/5} + \frac{4/3}{18/5} + \frac{2/3}{23/5} = 4.2011953$$

$$J_1 = \int_0^1 \frac{e^{2x} - P_4(x)}{x^{2/5}}dx$$

x	G(x)
0	0
1/6	0.0000743
2/6	0.0019118
3/6	0.0131271
4/6	0.0525669
5/6	0.1568599
1	0.3890561

$J_1 = 0.0654588$

$J = J_1 + J_2 = 4.2666541$

Version #2

$$\text{Let } t = x^{1/5} \quad x = t^5 \quad dx = 5t^4 dt \rightarrow \int_0^1 \frac{e^{2t^5}}{t^2}5t^4\,dt = \int_0^1 5t^2 e^{2t^5}\,dt$$

$$5t^2e^{2t^5} = 5t^2\left(1 + 2t^5 + \frac{(2t^5)^2}{2} + \frac{(2t^5)^3}{6} + \frac{(2t^5)^4}{24}\right) + 5t^2\left(e^{2t^5} - P_4(t)\right)$$

Extracted part: $\frac{5}{3} + \frac{10}{8} + \frac{10}{13} + \frac{10}{27} + \frac{5}{33} = 4.2077830$

$\int_0^1 \left(5t^2e^{2t^5} - P_4(t)\right)dt$ by n = 6 using Simpson's rule : 0.1105566

Total = 4.207783 + 0.1105566 = **4.3183396**

Note: This transformed equation is **no longer improper** and can be done by Simpson's rule itself
The result, however, is not good: 4.4863302
This is because the function rises sharply on the right:
$f(0) = 0$, $f(0.5) = 1.33$, and $f(1) = 36.95$.

Comparisons:

Version #1	**4.2666541**	by Derive:	**4.2651246**
Version #2	**4.3183396**	by Derive:	**4.2665408**
Simpson alone	**4.4863302**	by Derive:	**4.2665412 (based on series)**

note: for Derive the request in both cases was for an approximation to 8 digits

Comment: **Extraction helps in general**

Example #2: $\int_0^1 e^x\,dx = e - 1 = 1.7182818$

Simpson's Rule (4 points) - no extraction

$$\frac{1/4}{3}\left(0 + 4(1.2840254) + 2(1.6487213) + 4(2.1170000) + 2.7182818\right)$$

= **1.6349855** (Error = 0.0832963)

Simpson's Rule with extraction

$$e^x : P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$\int_0^1 P_4(x)\,dx = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 1.7166667$$

$$\int_0^1 e^x - P_4(x)\,dx = \frac{1/4}{3}\left(0 + 4(0.0000085) + 2(0.0002838) + 4(0.0022539) + 0.0099485\right)$$

$$= 0.0016305$$

I = 1.7166667 + 0.0016305 = **1.7182972** (error = 0.0000154)

Final note:

In general, transform $\int \frac{g(x)}{x^{p/q}}\,dx$ where $0 < \frac{p}{q} < 1$ by $t = x^{1/q}$