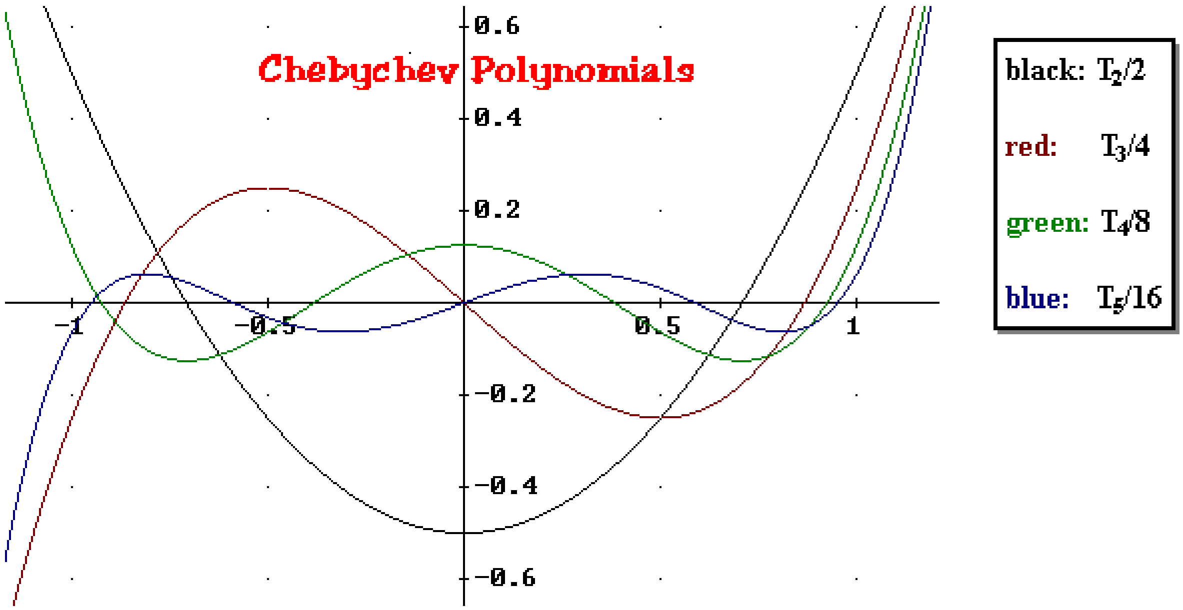
Chebychev Economization and Interpolation - Prof. Richard B. Goldstein

Goal: Approximate f(x) by $P_n(x)$ to minimize: $\max_{a \le x \le b} |f(x) - P_n(x)|$



Transform the interval [a, b] to [-1, 1] by x = (b-a)t/2+ (b+a)/2. It is best to make the error proportional to $T_{n+1}(x)$, the next higher degree Chebychev polynomial. The reason this is done is that of all the polynomials of the form $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$, the one that has smallest variation on the interval [-1, 1] is $T_n(x)/2^{n-1}$. Since $|T_n(x)| \le 1$ on [-1,1], $|T_n(x)/2^{n-1}| \le 1/2^{n-1}$.

Below are two methods which produce a first estimate to our goal. These could be further refined.

Goal: Approximate 2^x by $P_3(x)$ that minimizes absolute error on [0, 1]

Method: ECONOMIZATION

Start with the Taylor-Maclaurin Series expansion of $f(x) = 2^x$ about x = 0

$$2^x = 1 + (\ln 2)x + \frac{(\ln 2)^2}{2}x^2 + \frac{(\ln 2)^3}{6}x^3 + \frac{(\ln 2)^4}{24}x^4 + \frac{(\ln 2)^5}{120}x^5 + \frac{(\ln 2)^6}{720}x^6 + \ldots$$

This series has a truncation error of $\frac{2(\ln 2)^7}{5040}(1)^7 = 0.0000305$. Rewriting this series in terms of Chebychev Polynomials we find

$$2^x = T_0 + (\ln 2)T_1 + \frac{(\ln 2)^2}{2}\left(\frac{T_0 + T_2}{2}\right) + \frac{(\ln 2)^3}{6}\left(\frac{3T_1 + T_3}{4}\right) + \frac{(\ln 2)^4}{24}\left(\frac{3T_0 + 4T_2 + T_4}{8}\right)$$
$$+ \frac{(\ln 2)^5}{120}\left(\frac{10T_1 + 5T_3 + T_5}{16}\right) + \frac{(\ln 2)^6}{720}\left(\frac{10T_0 + 15T_2 + 6T_4 + T_6}{32}\right) \pm 0.0000305$$
$$2^x = 1.12376819\,T_0 + 0.73560861T_1 + 0.12499452\,T_2 + 0.014292701T_3$$
$$+ 0.0012311478\,T_4 + 0.0000833347T_5 + 0.0000048136T_6 \pm 0.0000305$$

Dropping the T_4, T_5, and T_6 terms adds to the error at most the absolute value of the coefficient multiplied by the T term:

Truncation Error ≈ 0.0000305 + 0.0000048136 + 0.0000833347 + 0.0012311478
Truncation Error ≈ 0.001350

Now, the series without the last three terms is rewritten in terms of powers:

$$2^x = 1.12376819(1) + 0.73560861(x) + 0.12499452(2x^2 - 1)$$
$$+ 0.014292701(4x^3 - 3x) \pm 0.001350$$

or

$2^x \approx 0.99877367 + 0.69273051x + 0.24998904x^2 + 0.057170803x^3$
with maximum absolute error $\leq$ 0.001350 on [0, 1]

Note: the actual maximum absolute error is 0.001336

Can we get more accuracy if we use the Taylor Series using 0.5 as the center of the interval? Yes! Start with the series:

$$2^{0.5t + 0.5} \approx 1.41421356 + 0.49012907t + 0.084932896t^2 + 0.0098118329t^3$$
$$+ 0.00085013054t^4 + 5.8926559 \times 10^{-5}\, t^5 + 3.4037315 \times 10^{-6}\, t^6$$
$$\approx 1.4569999T_0 + 0.49752478T_1 + 0.042893109T_2 + 0.0024713728T_3$$
$$+ 0.00010690452T_4 + 3.6829099 \times 10^{-6}\, T_5 + 1.0636609 \times 10^{-7}\, T_6$$

which has a truncation error of 1.685×10^{-7}. The resulting cubic after economization and substituting t = 2x - 1:

$2^x \approx 0.99989683 + 0.69638939x + 0.22451898x^2 + 0.079083929x^3$

This had a smaller maximum absolute error of 0.0001109.

Method: CHEBYCHEV INTERPOLATION

Use the roots of $T_{n+1}(x) = 0$, $\cos\left(\frac{(2k+1)\pi}{2n+2}\right), k = 0,1,\ldots,n$

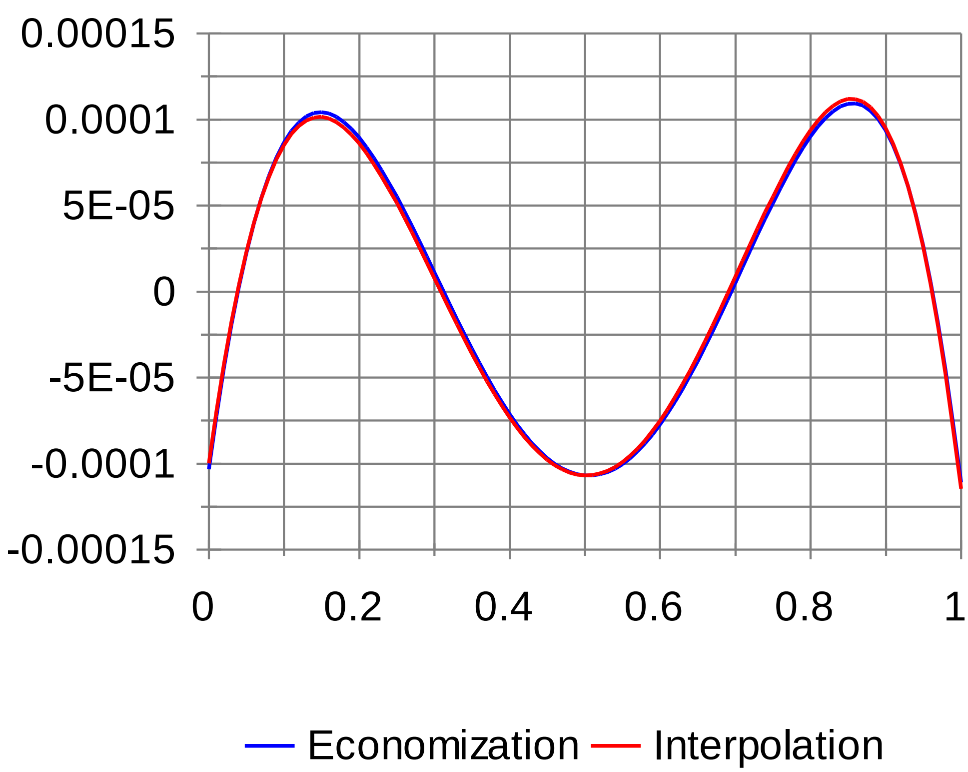
x_k	$\hat{x}_k = 0.5x_k + 0.5$	$2^{\hat{x}_k}$	1^{st} DD	2^{nd} DD	3^{rd} DD
$\cos(7\pi/8) = -0.92387953$	0.03806023	1.02673241			
			0.78279532		
$\cos(5\pi/8) = -0.38268343$	0.30865828	1.23855530		0.30666593	
			0.98313449		0.078967257
$\cos(3\pi/8) = +0.38268343$	0.69134172	1.61478458		0.37962216	
			1.23113462		
$\cos(\pi/8) = +0.92387953$	0.96193977	1.94792721			

This gives the following approximation:

$$2^x \approx 1.02673241 + 0.78279532(x - 0.03806023) + 0.30666593(x - 0.03806023)(x - 0.30865828) + 0.078967257(x - 0.03806023)(x - 0.30865828)(x - 0.69134172)$$

$$2^x \approx 0.99990029 + 0.69632477x + 0.22469316x^2 + 0.078967257x^3$$

The maximum absolute error on [0, 1] is: 0.0001145



Chebychev Polynomials - Prof. Richard B. Goldstein

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$
$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$

$$1 = T_0$$
$$x = T_1$$
$$x^2 = \frac{T_0 + T_2}{2}$$
$$x^3 = \frac{3T_1 + T_3}{4}$$
$$x^4 = \frac{3T_0 + 4T_2 + T_4}{8}$$
$$x^5 = \frac{10T_1 + 5T_3 + T_5}{16}$$
$$x^6 = \frac{10T_0 + 15T_2 + 6T_4 + T_6}{32}$$
$$x^7 = \frac{35T_1 + 21T_3 + 7T_5 + T_7}{64}$$
$$x^8 = \frac{35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8}{128}$$