Numerical Analysis – Prof. Richard B. Goldstein

Lecture One

Why study numerical analysis?

- Solution of Equations - Algebraic solutions are known for linear, quadratic, cubic and even fourth degree polynomial equations. It has been proved that solutions of fifth degree or higher degree polynomials cannot be found algebraically. They can, however, be found numerically by efficient and accurate methods. In addition other equations such as $f(x) = x - cox(x) = 0$ can only be solved numerically.

- Interpolation – In high school one may have learned how to use linear interpolation on a table in trigonometry or logarithms when calculators can now evaluate these functions very accurately. We have tables of functions such as the normal/Gaussian distribution where interpolation may be necessary. In addition, laboratory data, for example, may be measured every ten minutes or some other time interval but the values that are in between measurement times are unknown. Numerical analysis will show us how we can fit data with polynomials or rational polynomials to get more accurate approximations.

- Differentiation and Integration – How do we teach a machine how to estimate a derivative or evaluate an integral? Most applications require a numerical answer. Many definite integrals cannot be evaluated except by numerical methods or by a truncated power series.

- Functional Approximation – Computers and calculators only do simple arithmetic operations ($+ - \times \div$ ^) and circuits really only add 1's and 0's. How do use a computer to evaluate or approximate non-polynomial functions such as sin(x) or e^x? We can use polynomials, rational polynomials or other forms that require only simple mathematical operations to approximate these functions. How do we find the best representation?

Some Useful Theorems from Calculus

Intermediate Value Theorem

$f \in C[a,b], k$ between $f(a)$ and $f(b) \Rightarrow \exists c \in (a,b)$ s.t. $f(c) = k$ (written mathematically)

If f is a continuous function on the closed interval [a, b] and k is any number between f(a) and f(b), then there exists a c in the open interval (a, b) such that f(c) = k

This theorem helps us locate roots of a function after we have found an interval in which one end of the interval is negative and the other end is positive. The theorem is about the existence of a root and gives the user no information about how to find the root within that interval or whether the root is unique.

Mean Value Theorem

$$f \in C[a,b], f \text{ differentiable on}(a,b) \Rightarrow \exists c \in (a,b) \text{ s.t. } f'(c) = \frac{f(b) - f(a)}{b - a}$$

If f is a continuous function on the closed interval [a, b] and differentiable on the open interval (a, b), then there exists a point in the open interval, c, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The theorem tells that there is a "c" but does not tell us where c is or if it is unique.

Definition of Derivative at a Point

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Theoretically, a smaller and smaller h should give us a way of finding the derivative of a function at an arbitrarily chosen value of a. Sometimes that will work and other times the answer becomes worse at some point when h is too small. How do we choose an h or avoid this problem?

Examples

Intermediate Value Theorem

1.1 1 c. $f(x) = 2x\cos(2x) - (x-2)^2 = 0$ on [2, 3] and [3, 4]
[there is a change in sign]

Mean Value Theorem (or Rolle's Theorem as a special case when f(a) = f(b))

1.1 3 c. $f(x) = x\sin \pi x - (x-2)\ln x$ on [1, 2] [f(1) =f(2) = 0]

Estimating the Derivative

$$\boxed{\text{Estimating } f'(1) \text{ by } \frac{f(1+h)-f(1)}{h} \text{ for } f(x)=\ln(x) \text{ and } f(x)=\sqrt{x}}$$

h	ln(1+h)	chopped to 4 dec	est f ' (1)	rounded to 4 dec	est f ' (1)
A	B=ln(1+A)	C=int(10000*B)/10000	D=C/A	E=round(B,4)	F=E/A
1	0.69314718	0.6931	0.6931	0.6931	0.6931
0.5	0.40546511	0.4054	0.8108	0.4055	0.8110
0.2	0.18232156	0.1823	0.9115	0.1823	0.9115
0.05	0.04879016	0.0487	0.9740	0.0488	0.9760
0.02	0.01980263	0.0198	0.9900	0.0198	0.9900
0.01	0.00995033	0.0099	0.9900	0.0100	1.0000
0.005	0.00498754	0.0049	0.9800	0.0050	1.0000
0.002	0.00199800	0.0019	0.9500	0.0020	1.0000
0.001	0.00099950	0.0009	0.9000	0.0010	1.0000
0.0005	0.00049988	0.0004	0.8000	0.0005	1.0000
0.0002	0.00019998	0.0001	0.5000	0.0002	1.0000
0.0001	0.00010000	0	0.0000	0.0001	1.0000

h	sqrt(1+h)	chopped to 4 dec	est f ' (1)	rounded to 4 dec	est f ' (1)
A	B=sqrt(1+A)	C=int(10000*B)/10000	D=(C-1)/A	E=round(B,4)	F=(E-1)/A
1	1.41421356	1.4142	0.4142	1.4142	0.4142
0.5	1.22474487	1.2247	0.4494	1.2247	0.4494
0.2	1.09544512	1.0954	0.4770	1.0954	0.4770
0.05	1.02469508	1.0246	0.4920	1.0247	0.4940
0.02	1.00995049	1.0099	0.4950	1.0100	0.5000
0.01	1.00498756	1.0049	0.4900	1.0050	0.5000
0.005	1.00249688	1.0024	0.4800	1.0025	0.5000
0.002	1.00099950	1.0009	0.4500	1.0010	0.5000
0.001	1.00049988	1.0004	0.4000	1.0005	0.5000
0.0005	1.00024997	1.0002	0.4000	1.0002	0.4000
0.0002	1.00010000	1.0000	0.0000	1.0001	0.5000
0.0001	1.00005000	1.0000	0.0000	1.0000	0.0000

The results for both become better around h = 0.02 for chopping and then worse for smaller h.