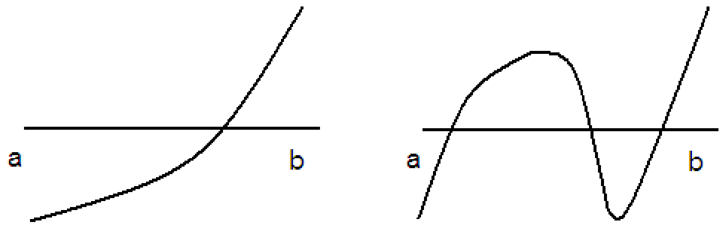
Bisection Method - Prof. Richard B. Goldstein



$f(a)<0$ $f(b)>0$ single root

$f(a)<0$ $f(b)>0$ multiple roots

If f(x) is a continuous function on the interval [a, b] and if the product $f(a)f(b) < 0$ then there is at least one root in the interval by the intermediate value theorem. That root might not be unique. The bisection method, which has been known since 1700 B.C., can be used to find at least one of the roots. It works by evaluating the function at both endpoints and in the middle and using the half of the interval which has a change in sign, and then repeats the process by narrowing the search interval where the root must appear by half with each step. After repeating a search 10 times the interval is one thousandth of its original size and after 20 times it is one millionth its original size. Although it is not the most efficient method, it is very robust and always finds a root.

BISECTION (Text Algorithm 2.1)

INPUT endpoints a, b; tolerance TOL; maximum number of iterations N_0

OUTPUT approximate solution p or message of failure

Step 1 Set $I = 1$;
$FA = f(a)$

Step 2 While $I \leq N_0$ do Steps 3-6.

Step 3 Set $p = a + (b - a)/2;$ (Compute p_i)
$FP = f(p);$

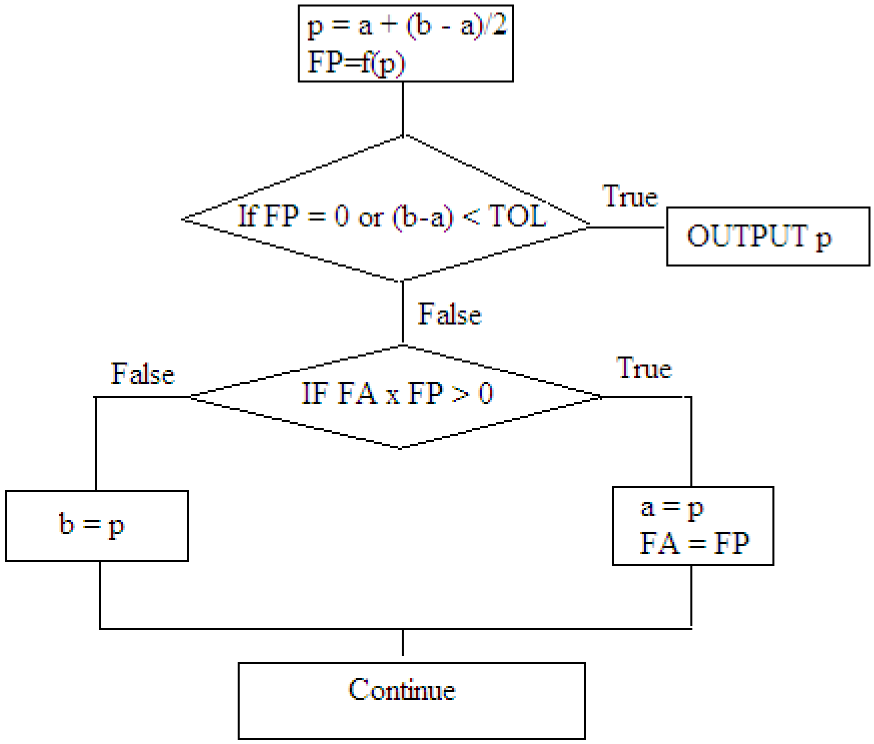
Step 4 If $FP = 0$ or $(b - a)/2 < TOL$ then
OUTPUT (p); (*Procedure completed successfully*.)
STOP

Step 5 Set $I = I + 1$

Step 6 If $FA \cdot FP > 0$ then set $a = p;$ (*Compute* a_i, b_i)
$FA = FP$
else set $b = p$

Step 7 OUTPUT ('Method failed after N_0 *iterations*, N_0 =', N_0)
(*The procedure was unsuccessful*)
STOP

Flowchart



Example

From our text: $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in [1, 2] since f(1) = -5 and f(2) = +14

We let a = 1, b = 2, and p = (a + b)/2 = 1.5 as the midpoint. Since f(1.5) = +2.375, we narrow our search to the left half interval [1, 1.5]. We repeat the process using a = 1, b = 1.5, and the new midpoint, p, as 1.25. Since f(1.25) = -1.797, this time we use the right half [1.25, 1.50] as our new search interval. Note that after two steps we have narrowed the search from the interval [1, 2] to [1.25, 1.5] which is one fourth of the original in width.

After 10 steps we have a root in [1.364258, 1.365234]. Using the midpoint of this interval, 1.364746, we have the root within ±0.000488 which is about 1/1000th of our original estimate of 0.5 ± 0.5, which is the midpoint of [0, 1] plus or minus the original half length of 0.5.

Our text shows a more complete table in section 2.1. Our first task is finding an appropriate starting interval [a, b]. Some problem will give you such an interval and others will have you find an appropriate starting interval.