Fixed Points / Iterations - Prof. Richard B. Goldstein

Roots can be found using an iterative procedure where $x_{n+1} = g(x_n)$. That is, starting with an initial estimate x_0, we find each successive new value x_1, x_2, x_3, … by plugging into an equation g(x). This function g(x) comes from replacing f(x) = 0 by an equation x = g(x) with the same solution or root.

A value of x = p where p = g(p) is called a **fixed point.** Some fixed points are stable where the sequence of values converges to that fixed point. Others fixed points are unstable and the sequence is divergent.

Consider the following iterations: [1] $x_{n+1} = 0.4x_n + 6$ [2] $x_{n+1} = 3x_n - 20$

For both iterations 10 is a fixed point. That is if one iterate is 10 then all of the following iterates are also 10. Suppose we let $x_0 = 5$. What iterates follow?

In [1] if $x_0 = 5$, then $x_1 = 8$, $x_2 = 9.2$, $x_3 = 9.68$, $x_4 = 9.872$, … a sequence converging to 10.
In [2] if $x_0 = 5$, then $x_1 = -5$, $x_2 = -35$, $x_3 = -125$, $x_4 = -395$, … a sequence diverging from 10.

What was the difference? Solving for p in [1] p = 0.4p + 6 gives 0.6p = 6 followed by p = 10 as does solving for p in [2] p = 3p – 20 which gives 2p = 20 followed by p = 10. Therefore, p = 10 is a fixed point for both. By testing various similar linear equations we find that we get convergence for the iteration $x_{n+1} = mx_n + b$ whenever $|m| < 1$ and divergence whenever $|m| > 1$.

Consider a simple quadratic $f(x) = x^2 - x - 6 = 0$. We can produce various g's for x = g(x). Here are 4 possibilities:

$g(x) = \sqrt{x+6}$ from $x^2 = x + 6$ and then take square roots

$g(x) = 1 + \frac{6}{x}$ from $x^2 = x + 6$ and then divide both sides by x

$g(x) = x^2 - 6$ from solving for the middle term

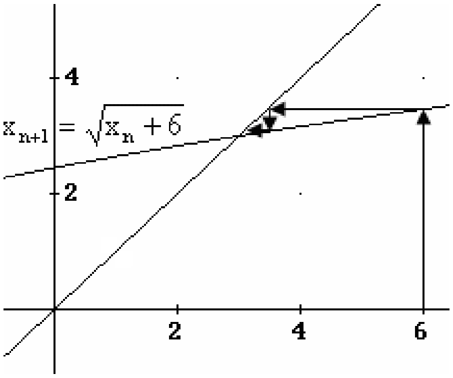
$g(x) = \frac{x^2+6}{2x-1}$ from Newton's method

Which of these converges? Note that the root x = 3 is a fixed point for all of these.

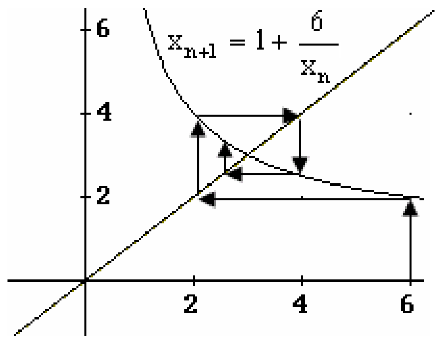
Fixed-Point Theorem

Let g be a continuous function on the closed interval [a, b] such that g(x) is bounded by [a, b]. Suppose, in addition, that the derivative of g exists on the open interval (a, b) and the absolute value of $g'(x)$ is bounded by a value of $k < 1$, then for any p_0 in [a, b] the sequence $p_{n+1} = g(p_n)$ converges to the unique fixed point p in [a,b]. The range of g is within [a, b] and the range for its derivative is within ±1.

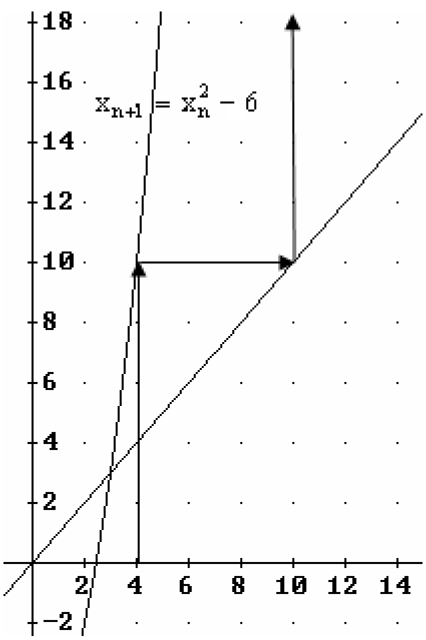
Iterative Solutions to $x^2 - x - 6 = 0$



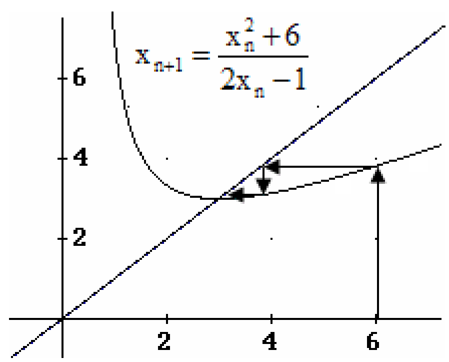
iter#	value
0	6.00000
1	3.46410
2	3.07638
3	3.01270
4	3.00212
5	3.00035



iter#	value
0	6.00000
1	2.00000
2	4.00000
3	2.50000
4	3.40000
5	2.76471
6	3.17021
7	2.89262
8	3.07425



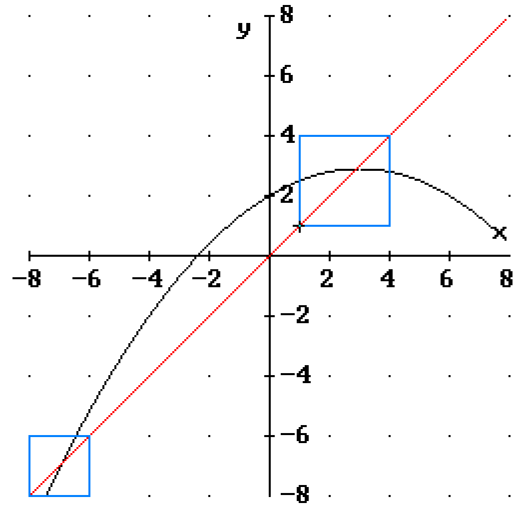
iter#	value
0	4
1	10
2	94
3	8830
4	77968894



iter#	value
0	6.000000
1	3.818182
2	3.100872
3	3.001956
4	3.000001

Fixed Point Example – Prof. Richard B. Goldstein

$$p_{n+1} = g(p_n) = -0.1p_n^2 + 0.6p_n + 2 \text{ on } [1, 4]$$



[1] Clearly the polynomial g(x) is continuous on [1, 4]

[2] g(1) = -0.1 + 0.6 + 2 = 2.5 ε [1, 4]
g(4) = -1.6 + 2.4 + 2 = 2.8 ε [1, 4]

Since $g'(x) = -0.2x + 0.6 = 0$ at $x = 3$ we consider this critical point:

g(3) = -0.9 +1.8 + 2 = 2.9 ε [1, 4]

g(x) has an absolute maximum of 2.9 and absolute minimum of 2.5 on [1, 4]. Therefore,
$1 \le g(x) \le 4 \;\forall x \varepsilon [1,4]$

[3] $g'(x) = -0.2x + 0.6$ exists on $(1,4)$

[4] $|g'(1)| = |-0.2 + 0.6| = 0.4 < 1$ $g'(x)$ is linear and monotonic. Let k = max{0.4,0.6} = 0.6
$|g'(4)| = |-0.8 + 0.2| = 0.6 < 1$

Starting with $p_0 = 2$ we find:

$p_1 = 2.8$
$p_2 = 2.896$
$p_3 = 2.8989184$
$p_4 = 2.898978251$
$p_5 = 2.8989794606$
$p_6 = 2.8989794851$
$p_7 = 2.8989794856 \; (= p_8 = p_9 = \cdots)$

Note:

Solve: $x = -0.1x^2 + 0.6x + 2 \Rightarrow x^2 + 4x - 20 = 0$
which yields two roots:

$$x = \frac{-4 \pm \sqrt{96}}{2} = -2 \pm 2\sqrt{6} = 2.898979486, -6.898979486$$

Try instead using the interval [-7, -6.8]

It is unstable since $|g'(-6.9)| = 1.98 > 1$