Newton-Raphson, Chords, Secant, and False Position

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NEWTON-RAPHSON

Tangent line at x_0 : $y - f(x_0) = f'(x_0)(x - x_0)$

Set y = 0 and find the value of x where the tangent line crosses the x axis:

$x - x_0 = - f'(x_0) / f(x_0) \rightarrow x = x_0 - f'(x_0) / f(x_0)$ call this value of x the next iterate x_1

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

Example: $f(x) = x^3 + 4x^2 - 10$, $f'(x) = 3x^2 + 8x$, $p_0 = 1.5$

$f(p_0) = 2.37500$, $f'(p_0) = 18.75000$, $p_1 = 1.50000 - (2.37500/18.75000) = 1.37333$

$f(p_1) = 0.13435$, $f'(p_1) = 16.64480$, $p_1 = 1.37333 - (0.13435/16.64480) = 1.36526$

$f(p_2) = 0.00053$, $f'(p_1) = 16.51392$, $p_1 = 1.36526 - (0.00053/16.51392) = 1.36523$

CHORDS

Either use $f'(p_0)$ or an estimate for λ in $p_n = p_{n-1} - \frac{f(p_{n-1})}{\lambda}$

Example: $f(x) = x^3 + 4x^2 - 10$, $\lambda = 20$, $p_0 = 1.5$

$f(p_0) = 2.37500$, $p_1 = 1.50000 - (2.37500/20) = 1.38125$

$f(p_1) = 0.26663$, $p_1 = 1.38125 - (0.26663/20) = 1.36792$

$f(p_2) = 0.04446$, $p_1 = 1.36792 - (0.04446/20) = 1.36570$

is converging but more slowly than Newton-Raphson

SECANT

Replace the slope of the tangent, $f'(p_{n-1})$, by the slope of the secant $(f(p_{n-1}) - f(p_{n-2}))/(p_{n-1} - p_{n-2})$

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

Example: $f(x) = x^3 + 4x^2 - 10$, $p_0 = 1$, $p_1 = 2$, $f(p_0) = -5.00000$, $f(p_1) = 14.00000$

$p_2 = 2.00000 - 14.00000(2.00000-1.00000)/(14.00000 -(-5.00000)) = 1.26316$, $f(p_2) = -1.60227$

$p_3 = 1.26316 -(-1.60227)(1.26316-2.00000)/(-1.60227 -14.00000) = 1.33883$, $f(p_3) = -0.43036$

$p_4 = 1.33883 -(-0.43036)(1.33883-1.26316)/(-0.43036 -(-1.60227)) = 1.36662$, $f(p_3) = 0.02291$

$p_5 = 1.36662 -(0.02291)(1.36662-1.33883)/(0.02291-(-0.43036)) = 1.36521$, $f(p_3) = -0.00030$

FALSE POSITION

Uses the idea of bisection – keeping **a** and **b** such that **f(a)** and **f(b)** are opposite in sign.

$$p_n = a - \frac{f(a)(b - a)}{f(b) - f(a)}$$

Example: $f(x) = x^3 + 4x^2 - 10$, $a = 1$, $b = 2$, $f(a) = -5.00000$, $f(b) = 14.00000$

$p_1 = 2.00000 - 14.00000(2.00000-1.00000)/(14.00000 -(-5.00000)) = 1.26316$, $f(p_1) = -1.60227$
since $f(p_1)$ is negative and f(b) is positive we replace **a** by p_1

$p_2 = 1.26316 - (-1.60227)(2.00000-1.26316)/(14.00000-(-1.60227)) = 1.33883$, $f(p_2) = -0.43036$
since $f(p_2)$ is negative and f(b) is positive we replace **a** by p_2

$p_3 = 1.33883 - (-0.43036)(2.00000-1.33883)/(14.00000-(-0.43036)) = 1.35855$, $f(p_3) = -0.11001$
since $f(p_3)$ is negative and f(b) is positive we replace **a** by p_3

$p_4 = 1.35855 - (-0.11001)(2.00000-1.35855)/(14.00000-(-0.11001)) = 1.36355$, $f(p_3) = -0.02776$
since $f(p_4)$ is negative and f(b) is positive we replace **a** by p_4

In the false position method it is not unusual to just replace only one side for all or almost of the steps rather than sometimes **a** and other times **b.**

MTH/CSC – Prof. Richard B. Goldstein – Finding Roots

Let $\varepsilon_n = |p_n - p|$ and $\varepsilon_{n+1} \approx \lambda(\varepsilon_n)^\alpha$ where α = order of convergence

Method	Formula	α	λ
Bisection *	$p_n = \frac{a+b}{2}$	1	0.5
Fixed Point	$p_n = g(p_{n-1})$	1	$g'(p)$
Newton-Raphson	$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$	2	$\left\|\frac{f''(p)}{2f'(p)}\right\|$
Chords	$p_n = p_{n-1} - \frac{f(p_{n-1})}{k},\ k = f'(p_0)$	1	$\left\|1 - \frac{f'(p)}{k}\right\|$
Secant	$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$	1.618	$\left\|\frac{f''(p)}{2f'(p)}\right\|$
False Position *	$p_n = a - \frac{f(a)(b-a)}{f(b) - f(a)}$	1	varies
Bairstow	quadratic factors	2	

* requires $f(a)f(b) < 0$ all the time

Multiple Roots

$f(x) = (x - p)^m h(x)$

Newton is reduced to α =1 and λ = (m – 1)/m. Quadratic convergence is restored if

[a] m known: $p_n = p_{n-1} - m\frac{f(p_{n-1})}{f'(p_{n-1})}$

[b] m unknown: let $u(x) = f(x)/f'(x)$ then

$$p_n = p_{n-1} - \frac{u(p_{n-1})}{u'(p_{n-1})} = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{[f'(p_{n-1})]^2 - f(p_{n-1})f''(p_{n-1})}$$

Accelerating Convergence

$$\frac{p_{n+1}-p}{p_n-p} \approx \frac{p_{n+2}-p}{p_{n+1}-p} \Rightarrow p = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$$

Steffensens' Method

$p_1 = g(p_0)$

$p_2 = g(p_1)$

$$p_0' = p_0 - \frac{(p_1-p_0)^2}{p_2-2p_1+p_0}$$

repeat for p_1', p_2', p_0'', etc.

Comparisons – number of iterations for $x^3 + 4x^2 - 10 = 0$

Method	Initial Values	error less than		
		10^{-3}	10^{-6}	10^{-9}
Bisection	a = 1, b = 2	9	19	29
Fixed Point	$p_0 = 1.5,\ g(x) = \frac{1}{2}\sqrt{10-x^3}$	8	18	29
Steffensen's	*same as above*, $p_0 = 1.5$ $p_0^{k+1} = p_0^k - \frac{(p_1^k - p_0^k)^2}{p_2^k - 2p_1^k + p_0^k}$	4	5	7
Newton-Raphson	$p_0 = 1.5$	2	3	3
Chords	$p_0 = 1.5$	2	6	9
Secant	$p_0 = 1,\ p_1 = 2$	4	5	6
False Position	a = 1, b = 2	4	9	15

Clearly, the Newton-Raphson method requires the fewest steps for a good starting point. The next most efficient is the secant method and requires only one function evaluation at each step to the two function evaluations needed in Newton-Raphson. The bisection and fixed point methods required the most steps.