INTERPOLATION AND POLYNOMIAL APPROXIMATION

Prof. Richard B. Goldstein

Karl Weierstrass was the first to prove that there is always a polynomial that can be used to approximate any function f(x) on a closed interval within a band of $\pm\varepsilon$. In addition, since polynomials are very easy to evaluate on a computer, they become the prime form to use for representing a function. We will consider various methods and will also consider the use of a rational polynomial.

In the general problem one is given n + 1 points $(x_0, f_0), (x_1, f_1), \ldots, (x_n, f_n)$. Just as two points determine a unique line, n + 1 points determine a unique n^{th} degree polynomial. Several mathematicians found different ways to find that n^{th} degree polynomial. We will consider the methods of Lagrange, Neville, and Newton.

- Lagrange considered n^{th} degree polynomials $L_{n,k}(x)$ which take on the value 1 at $x = x_k$ and the value 0 at any of the other values of x, namely $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$
  We then set $P_n(x)$, our polynomial fit, to be a linear combination of these n^{th} degree polynomials $L_{n,k}(x)$ with weights of $f_0, f_1, \ldots, f_n$. The resulting $P_n(x)$ will then correctly taken on the value f_k at $x = x_k$ for k ranging from 0 to n.

- Neville found a way to start with a table of the n + 1 points and their function values and come up with the same value as Lagrange, $P_n(x)$, at any arbitrary point x. His method never actually finds the polynomial, just its value at x. This is done by first considering all linear equations through pairs of consecutive points, then all quadratic equations through all sets of three consecutive points, then all cubic equations through all sets of four consecutive points, until finally considering the unique polynomial value through all n + 1 points. This is very convenient for programmers since it is done with a double array and need not use the variable, x.

- Newton's divided difference method finds a simple tabular way of finding $P_n(x)$ in the form $A + B(x - x_0) + C(x - x_0)(x - x_1) + D(x - x_0)(x - x_1)(x - x_2) + \ldots$ The values of the coefficients A, B, C, D, … etc. are found from this table. Newton's method greatly simplifies to a simple difference table when the n + 1 points are equally spaced.

- Two other special cases are considered. Hermite found a unique polynomial that would be given through the n + 1 points along with knowledge of the first derivatives, $f'(x_k)$, at each point. Although not in our text, many other texts give an algorithm for finding a unique rational polynomial P(x)/Q(x) through the n + 1 points.

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Weierstrass Approximation Theorem

$$f \in C[a,b].\ \text{For each } \varepsilon > 0,\ \exists P_n(x) \text{ s.t. } \left| f(x) - P_n(x) \right| < \varepsilon\ \forall x \in [a,b]$$

Given Information: n + 1 points: $x_0, f_0; x_1, f_1; \ldots; x_n, f_n$ where $f_k = f(x_k)$

Lagrange Polynomials

$$P_n(x) = f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + \cdots + f(x_n)L_{n,n}(x) \quad \text{where}$$

$$L_{n,k} = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$$

Error Formula: $$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Neville's Formula

x_0	$f(x_0) = P_0 = Q_{0,0}$				
x_1	$f(x_1) = P_1 = Q_{1,0}$	$P_{0,1} = Q_{1,1}$			
x_2	$f(x_2) = P_2 = Q_{2,0}$	$P_{1,2} = Q_{2,1}$	$P_{0,1,2} = Q_{2,2}$		
x_3	$f(x_3) = P_3 = Q_{3,0}$	$P_{2,3} = Q_{3,1}$	$P_{1,2,3} = Q_{3,2}$	$P_{0,1,2,3} = Q_{3,3}$	
x_4	$f(x_4) = P_4 = Q_{4,0}$	$P_{3,4} = Q_{4,1}$	$P_{2,3,4} = Q_{4,2}$	$P_{1,2,3,4} = Q_{4,3}$	$P_{0,1,2,3,4} = Q_{4,4}$

where $$Q_{i,j} = \frac{(x - x_{i-j})Q_{i,j-1} - (x-x_i)Q_{i-1,j-1}}{(x_i - x_{i-j})} \quad \text{Error} \approx \left| Q_{n,n} - Q_{n-1,n-1} \right|$$

Newton's Divided-Difference Formula

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x-x_0)\cdots(x-x_{k-1})$$

x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

where $$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i},\ f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}, \text{ etc.}$$

Newton Forward Difference Formula (equally spaced points only)

$$P_n(x) = f(x_0) + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0 + \cdots + \frac{s^{(k)}}{k!}\Delta^k f_0$$

where $x = x_0 + sh,\ x_i = x_0 + ih$ (equally spaced points)

$s^{(k)} = s(s-1)(s-2)\cdots(s-k+1)$ and

$\Delta f_0 = f_1 - f_0$

$\Delta^2 f_0 = \Delta(f_1 - f_0) = f_2 - 2f_1 + f_0$

$\Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0$

Newton Backward Difference Formula (equally spaced points only)

$$P_n(x) = f(x_n) + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_n + \cdots$$

$\nabla f_n = f_n - f_{n-1},\ \nabla^2 f_n = f_n - 2f_{n-1} + f_{n-1}, \ldots$

Hermite Interpolation

Given Information: n + 1 points: $x_0, f_0, f_0'; x_1, f_1, f_1'; \ldots; x_n, f_n, f_n'$

where $f_k = f(x_k)$ and $f_k' = f'(x_k)$

$$H_{2n+1}(x) = f[z_0] + \sum_{k=1}^{2n+1} f[z_0,\ldots,z_k](x-z_0)(x-z_1)\cdots(x-z_{k-1})$$

$z_0 = x_0 \quad f[z_0]=f(x_0)$

$f[z_0,z_1] = f'(x_0)$

$z_1 = x_0 \quad f[z_1]=f(x_0)$

$f[z_0,z_1,z_2] = \frac{f[z_1,z_2]-f[z_0,z_1]}{z_2 - z_0}$

$f[z_1,z_2] = \frac{f[z_2]-f[z_1]}{z_2 - z_1}$

$z_2 = x_1 \quad f[z_2]=f(x_1)$

$f[z_1,z_2,z_3] = \frac{f[z_2,z_3]-f[z_1,z_2]}{z_3 - z_1}$

$f[z_2,z_3] = f'(x_1)$

$z_3 = x_1 \quad f[z_3]=f(x_1)$

$f[z_2,z_3,z_4] = \frac{f[z_3,z_4]-f[z_2,z_3]}{z_3 - z_1}$

$f[z_3,z_4] = \frac{f[z_4]-f[z_3]}{z_4 - z_3}$

$z_4 = x_2 \quad f[z_4]=f(x_2)$

$f[z_3,z_4,z_5] = \frac{f[z_4,z_5]-f[z_3,z_4]}{z_5 - z_3}$

$f[z_4,z_5] = f'(x_2)$

$z_5 = x_2 \quad f[z_5]=f(x_2)$

INTERPOLATION EXAMPLES – Prof. Richard B. Goldstein

Given points: (1, 6), (2, 4), (3, 3) and (5, 2) Estimate f(4)

Lagrange Polynomials

$$P_3(x) = 6\frac{(x-2)(x-3)(x-5)}{(1-2)(1-3)(1-5)} + 4\frac{(x-1)(x-3)(x-5)}{(2-1)(2-3)(2-5)} +$$
$$3\frac{(x-1)(x-2)(x-5)}{(3-1)(3-2)(3-5)} + 2\frac{(x-1)(x-2)(x-3)}{(5-1)(5-2)(5-3)}$$

$$P_3(x) = 6\left(\frac{6}{24}\right) + 4\left(\frac{-24}{24}\right) + 3\left(\frac{36}{24}\right) + 2\left(\frac{6}{24}\right) = \frac{60}{24} = 2.5$$ **(actual is 2.4)**

Neville's Method

$$Q_{i,j} = \frac{(x - x_{i-j})Q_{i,j-1} - (x - x_i)Q_{i-1,j-1}}{x_i - x_{i-j}}$$

Error estimate = |2.5 – 3| = 0.5

$x_0 = 1$	$Q_{0,0} = 6$			
$x_1 = 2$	$Q_{1,0} = 4$	$Q_{1,1} = 0$		
$x_2 = 3$	$Q_{2,0} = 3$	$Q_{2,1} = 2$	$Q_{2,2} = 3$	
$x_3 = 5$	$Q_{3,0} = 2$	$Q_{2,1} = 2.5$	$Q_{3,2} = 2.333333$	$Q_{3,3} = 2.5$

Divided Differences

x_i	$F[x_i]$	$F[x_i, x_{i+1}]$	$F[x_i, x_{i+1}, x_{i+2}]$	$F[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
1	6			
		-2		
2	4		0.5	
		-1		-0.083333333
3	3		0.16666666	
		-0.5		
5	2			

$P_3(4) = 6 - 2(4 - 1) + 0.5(4 - 1)(4 - 2) - 0.08333333(4 - 1)(4 - 2)(4 - 3) = 2.5$

Error Term

Using $f(x) = \frac{12}{x+1}$, error $\le \frac{f^4(c)}{4!}(x-1)(x-2)(x-3)(x-5)$

Since $f^4 = 288(c+1)^{-5}$ for $1 \le c \le 5$, error $\le \left|\frac{9}{24}(3)(2)(1)(-1)\right| = 2.25$

Differences

1	6				
		-2			
2	4		1		
		-1		-0.6	
3	3		0.4		0.4
		-0.6		-0.2	
4	2.4		0.2		
		-0.4			
5	2				

Forward $x = 3.8 = x_0 + sh = 1 + s(1) \rightarrow s = 2.8$

$$P_4(3.8) = 6 + 2.8(-2) + \frac{2.8(1.8)}{2}(1) + \frac{2.8(1.8)(0.8)}{6}(-0.6) + \frac{2.8(1.8)(0.8)(-0.2)}{24}(0.4)$$

$$P_4(3.8) = 6 - 5.6 + 2.52 - 0.4032 - 0.01344 = 2.50336$$

Backward $x = 3.8 = x_n + sh = 5 + s(1) \rightarrow s = -1.2$

$$P_4(3.8) = 2 - 1.2(-0.4) + \frac{-1.2(-0.2)}{2}(0.2) + \frac{-1.2(-0.2)(0.8)}{6}(-0.2) + \frac{-1.2(1-0.2)(0.8)(1.8)}{24}(0.4)$$

$$P_4(3.8) = 2 + 0.48 - 0.0064 + 0.00576 = 2.50336$$

Hermite

Given Information: 2 pts: $f(3) = 3,\ f'(3) = -0.75,\ f(5) = 2,\ f'(5) = -0.3\overline{3}$ using $f(x) = \dfrac{12}{x+1}$

x	f(x)	1st DD or Derivative	2nd DD	3rd DD
3	3			
		-0.75		
3	3		0.125	
		-0.5		-0.0208333…
5	2		0.08333…	
		-0.333…		
5	2			

$P_3(x) = 3 - 0.75(x - 3) + 0.125(x - 3)^2 - 0.0208333\ldots(x - 3)^2(x - 5)$

$P_3(4) = 3 - 0.75 + 0.125 + 0.0208333\ldots = 2.3958333\ldots$ (actual is 2.4)