Numerical Differentiation – Prof. Richard B. Goldstein

Differentiation

3 – POINTS: $(x_{-1}, f_{-1}), (x_0, f_0)$, and (x_1, f_1) where $h = x_1 - x_0 = x_0 - x_{-1}$

$$f_0{}' = \frac{f_1 - f_{-1}}{2h} + O(h^2) \qquad f_0{}'' = \frac{f_1 - 2f_0 + f_{-1}}{h^2} + O(h^2)$$

Non-Central Left: $f_0{}' = \frac{1}{h}[-1.5f(x_0) + 2f(x_1) - 0.5f(x_2)] + O(h^2)$

Right: $f_n{}' = \frac{1}{h}[0.5f(x_{n-2}) - 2f(x_{n-1}) + 1.5f(x_n)] + O(h^2)$

5 – POINTS: $(x_{-2}, f_{-2}), (x_{-1}, f_{-1}), (x_0, f_0), (x_1, f_1)$, and (x_2, f_2)

$$f_0{}' = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + O(h^4)$$

$$f_0{}'' = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} + O(h^4)$$

$$f_0{}^{(3)} = \frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3} + O(h^4)$$

$$f_0{}^{(4)} = \frac{f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}}{h^4} + O(h^4)$$

RICHARDSON'S EXTRAPOLATION:

$$N_1(h) = \frac{f_1 - f_{-1}}{2h}$$

$$N_1\left(\tfrac{h}{2}\right) = \frac{f_{0.5} - f_{-0.5}}{h}$$

$$N_1\left(\tfrac{h}{4}\right) = \frac{f_{0.25} - f_{-0.25}}{0.5h}$$

$$N_2(h) = N_1\left(\tfrac{h}{2}\right) + \frac{N_1\left(\tfrac{h}{2}\right) - N_1(h)}{3}$$

$$N_2\left(\tfrac{h}{2}\right) = N_1\left(\tfrac{h}{4}\right) + \frac{N_1\left(\tfrac{h}{4}\right) - N_1\left(\tfrac{h}{2}\right)}{3}$$

$$N_3(h) = N_2\left(\tfrac{h}{2}\right) + \frac{N_2\left(\tfrac{h}{2}\right) - N_2(h)}{15}$$

PARTIAL DIFFERENTIATION:

$$f_x(x_0, y_0) = \frac{f_{1,0} - f_{-1,0}}{2h} + O(h^2) \quad \text{and} \quad f_x(x_0, y_0) = \frac{f_{0,1} - f_{0,-1}}{2k} + O(k^2)$$

$$f_{xx}(x_0, y_0) = \frac{f_{1,0} - 2f_{0,0} + f_{-1,0}}{h^2} + O(h^2)$$

$$f_{xy}(x_0, y_0) = \frac{f_{1,1} - f_{-1,1} + f_{-1,-1} - f_{1,-1}}{4hk} + O(hk)$$

where $h = x_1 - x_0 = x_0 - x_{-1}$ and $k = y_1 - y_0 = y_0 - y_{-1}$