Numerical Differentiation - Prof. Richard B. Goldstein

Differentiation

3 – POINTS: x = a – h, a, and a + h

$f'(a) \approx [f(a + h) – f(a – h)]/2h$

$f''(a) \approx [f(a + h) – 2f(a) + f(a – h)]/h^2$

Non-Central Left: $f'(a) \approx [-1.5f(a) + 2f(a + h) – 0.5f(a + 2h)]/h$

Right: $f'(a) \approx [0.5f(a) - 2f(a - h) + 1.5f(a - 2h)]/h$

5 – POINTS: x = a – 2h, a – h, a, a + h, and a + 2h

$f'(a) \approx [-f(a + 2h) + 8f(a + h) – 8f(a – h) + f(a – 2h)]/12h$

$f''(a) \approx [-f(a + 2h) + 16f(a + h) – 30f(a) + 16f(a - h) - f(a – 2h)]/12h^2$

RICHARDSON'S EXTRAPOLATION:

$N_1(h) = [f(a + h) – f(a – h)]/2h$

$N_1(0.5h) = [f(a + 0.5h) – f(a – 0.5h)]/h$

$N_1(0.25h) = [f(a + 0.25h) – f(a – 0.25h)]/0.5h$

$N_1(0.125h) = [f(a + 0.125h) – f(a – 0.125h)]/0.25h$

$N_2(h) = [4N_1(0.5h) – N_1(h)]/3$

$N_2(0.5h) = [4N_1(0.25h) – N_1(0.5h)]/3$

$N_2(0.25h) = [4N_1(0.125h) – N_1(0.25h)]/3$

$N_3(h) = [16N_2(0.5h) – N_2(h)]/15$

$N_3(0.5h) = [16N_2(0.25h) – N_2(0.5h)]/15$

$N_4(h) = [64N_3(0.5h) – N_3(h)]/63$

Example:

Let f(x) = ln x. Then f ′(x) = 1/x, f ′(2) = 0.5 and f ′′(x) = $-1/x^2$, f ′′(2) = -0.25

Letting h = 0.1 in the **3-point formulas**

f ′(2) ≈ [ln(2.1) – ln(1.9)]/0.2 = [0.741937 – 0.641854]/0.2 = 0.500417

f ′′(2) ≈ [ln(2.1) – 2ln(2) + ln(1.9)]/0.1^2

= [0.741937 – 2(0.693147) + 0.641854]/0.01 = -0.2503

Letting h = 0.1 in the **5-point formulas**

f ′(2) ≈ [-ln(2.2) + 8ln(2.1) – 8ln(1.9) + ln(1.8)]/1.2

= [-0.788457 + 8(0.741937) – 8(0.641854) + 0.587787]/1.2 = 0.499995

f ′′(2) ≈ [-ln(2.2) + 16ln(2.1) – 30ln(2) + 16ln(1.9) – ln(1.8)]/0.12

= [0.788457+16(0.741937)–30(0.693147)+ 16(0.641854)–0.587787]/0.12

= -0.029998/1.2 = -0.249983

Richardson's Extrapolation (now with 7 decimal digits)

$N_1(0.1)$ = 0.5004173 (from above 3-pt formula)

$N_1(0.05)$ = [ln(2.05) – ln(1.95)]/0.1 ≈ [0.7178398 – 0.6678294]/0.1 = 0.5001042

$N_1(0.025)$ = [ln(2.025)–ln(1.975)]/0.05 ≈ [0.7055697–0.6805684]/0.05 = 0.5000260

$N_1(0.0125)$ = [ln(2.0125)–ln(1.9875)]/0.025 ≈ [0.6993778–0.6868776]/0.025 = 0.5000065

$N_2(0.1)$ = [4(0.5001042) – 0.5004173]/3 = 0.4999998

$N_2(0.05)$ = [4(0.500026) – 0.5001042]/3 = 0.4999999

$N_2(0.025)$ = [4(0.5000065) – 0.5000260]/3 = 0.5000000

$N_3(0.1)$ = [16(0.4999999) – 0.4999998]/15 = 0.4999999

$N_3(0.05)$ = [16(0.5000000) – 0.4999999]/15 = 0.5000000

$N_4(0.1)$ = [64(0.5000000) – 0.4999999]/63 = 0.5000000